Trees are graceful.

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Abstract. We answer the question of Ringel-Kotzig about graceful labelings of trees.

Definition. *For every $N \geq 1$, a N-tree is an undirected connected acyclic graph $G = (V, E)$ with $N = |V|$ vertices. Hence G has $N-1 = |E|$ edges denoted $\{x, y\}$. For every $M \geq N$, a M-labeling of G is an injection σ from V to $[0, \ldots, M-1]$ such that the coloring of edges with $\sigma(\{x, y\}) = |\sigma(x) - \sigma(y)|$ is an injection from E to $[1, \ldots, M-1]$. The N-tree G is said graceful when it has a N-labeling.*

Observe that a N-labeling of a N-tree and the corresponding coloring of edges are both bijective.

Theorem. *Every N-tree is graceful.*

We will prove this statement by induction on N. We need more definitions.

Definition. *Let G be a N-tree.*
A vertex x is a leaf of G when it belongs to exactly one edge. Otherwise, x is a link.
For $M > N$, a M-labeling of G is z-free for some z in $[1, \ldots, M-1]$, when no vertex has the label z and no edge has the color z.

Observe that every N-labeling of a tree is a N free $(N+1)$-labeling of this tree.

Proposition. *For every $N \geq 1$ and every N-tree G,*
(g_N) : there exists a N-labeling σ of G.
(f_N) : for every vertex x of G, there exists a z-free $(N+1)$-labeling ρ of G with $\rho(x) = 0$.